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Study of the double mathematical pendulum: II. Investigation of exponentially small homoclinic intersections

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Abstract
We consider the double mathematical pendulum in the limit when the ratio of pendulum masses is close to zero and the ratio of pendulum lengths is close to infinity. We found that the limit system has a hyperbolic periodic trajectory, whose invariant manifolds intersect transversally and the intersections are exponentially small. In this case we obtain an asymptotic formula of the homoclinic invariant for the limit system.

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1. Introduction

In this paper we continue the investigation of the double mathematical pendulum started in [9]. The double mathematical pendulum is a classical example of a Hamiltonian system with two degrees of freedom. It consists of two masses m_1 and m_2 attached to consequently joined arms of lengths l_1 and l_2, respectively, the upper end of the first arm being fixed, and the whole system being subjected to the action of the constant gravity acceleration, g. We take the angles ϕ_1 and ϕ_2 of deviation of the arms from the vertical axis as the coordinates and define new parameters of the system δ, ε, ν in the same manner as in [9]:

$$\delta = \frac{m_2}{m_1} \qquad \varepsilon = \frac{l_2}{l_1} \qquad \nu = \sqrt{\frac{E}{2m_1 g l_1}}$$

where E is the energy of the system.

The kinetic energy and the potential energy are of the form

$$T = \tfrac{1}{2}(1+\delta)\dot{\phi}_1^2 + \tfrac{1}{2}\delta\varepsilon^2\dot{\phi}_2^2 + \delta\varepsilon\cos(\phi_1-\phi_2)\dot{\phi}_1\dot{\phi}_2$$
$$U = (1+\delta)(1-\cos\phi_1) + \delta\varepsilon(1-\cos\phi_2).$$

Using the Hamiltonian formalism, we introduce the generalized momenta $p_i = \frac{\partial L}{\partial \dot{\phi}_i}$, $i = 1, 2$, where $L = T - U$ is the Lagrangian of the system

$$\begin{aligned} p_1 &= (1+\delta)\dot{\phi}_1 + \delta\varepsilon\cos(\phi_1 - \phi_2)\dot{\phi}_2 \\ p_2 &= \delta\varepsilon^2\dot{\phi}_2 + \delta\varepsilon\cos(\phi_1 - \phi_2)\dot{\phi}_1. \end{aligned} \tag{1.1}$$

Then the double mathematical pendulum is described by the Hamilton equations

$$\dot{\phi}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial \phi_i} \qquad i = 1, 2 \tag{1.2}$$

with the Hamiltonian H:

$$\begin{aligned} H = {} & \frac{1}{2\left(1 + \delta\sin^2(\phi_1 - \phi_2)\right)}\left(p_1^2 - 2\frac{p_1 p_2\cos(\phi_1 - \phi_2)}{\varepsilon} + \frac{p_2^2(1+\delta)}{\delta\varepsilon^2}\right) \\ & + (1+\delta)(1 - \cos\phi_1) + \delta\varepsilon(1 - \cos\phi_2). \end{aligned} \tag{1.3}$$

Since the system (1.2) is conservative, the motion occurs on the energy surface

$$H(\phi_1, \phi_2, p_1, p_2; \delta, \varepsilon) = 2\nu^2$$

and the value of the relative energy ν is treated as an additional parameter.

Thus, the system (1.2) is considered to be depending on three parameters. The space of natural parameters $(\delta, \varepsilon, \nu)$ is $\mathbb{R}_+^3$, where $\mathbb{R}_+ = (0, +\infty)$, and it is noncompact. Let us compactify it by adding limit values $\{0, +\infty\}$ to each copy of $\mathbb{R}_+$. So we can consider topologically this compactified parameters space as a cube.

As it was mentioned earlier the double mathematical pendulum had been investigated by the author in [9]. In that paper we studied numerically the system in terms of the Poincaré sections. The numerical method applied in that investigation consists of several steps. The first one is the finding of hyperbolic periodic points for the Poincaré map. The approximate positions of some hyperbolic periodic points were obtained visually from the pictures of the phase portrait (figure 1), and their precise coordinates were calculated by use of the Newton method. The second step of the method is the construction of separatrices. The pictures of the corresponding stable and unstable manifolds were drawn. By using the Tailor expansion near the hyperbolic points we obtained approximations of the separatrices in small neighbourhoods of that points (figure 2). After that we continue with them out of the neighbourhoods by applying the Poincaré map. The last step is to prove the existence of homoclinic transversal intersections. Comparing the constructed separatrices, positions of some homoclinic points and their homoclinic invariants were calculated. From the nonnullity of the homoclinic invariant followed the nonintegrability of the system. We applied this scheme to three chosen sets of system parameters and values of the energy and proved the nonintegrability of the system for these values.

It is to be noted that we used a modified definition of the nonintegrability because of the energy was considered as an additional parameter. More precisely, the system is said to be *integrable* for the given values of the parameters δ, ε, ν if there exists a neighbourhood of the energy surface $H^{-1}(\nu)$, where there are two independent integrals of motion. Otherwise it is called *nonintegrable* for δ, ε, ν.

From the data obtained in numerical experiments we stated [9], the main conjecture is that *the double mathematical pendulum is nonintegrable for all nontrivial (not equal to zero or infinity) values of the parameters*.

To justify this conjecture we suggest to prove that the system has on every energy level $H^{-1}(\nu)$ and for any values of two other parameters a hyperbolic periodic trajectory, whose separatrices intersect transversally. The method proposed in [9] gives an instrument

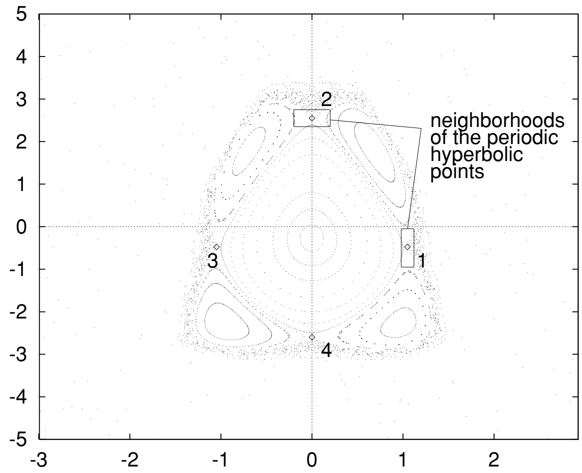


Figure 1. The Poincaré section, corresponding to ($\delta = 0.01$, $\varepsilon = \sqrt{5}$, $\nu = 1.16$).

for proving the nonintegrability of the system for some fixed $(\delta, \varepsilon, \nu)$. But as it follows from the stability of hyperbolic objects under small perturbations [12], if the system (1.2) is nonintegrable for some $(\delta_0, \varepsilon_0, \nu_0)$, there is a neighbourhood of this point in the space of parameters where the system is nonintegrable too. If the space of parameters were compact, we would try to cover it by a finite number of such neighbourhoods, where (1.2) is nonintegrable. But it is not so. That is why we have to apply asymptotic methods to investigate the system in the limit when the values of the parameters are close to the limit ones. Thus, a strategy of proving the conjecture can be based on the idea of combination of asymptotic and numerical methods. By using different asymptotic techniques we try to pick up a vicinity of the parameters space boundary, where the transversal homoclinic intersections exist. Then one should divide the supplement of the vicinity by a finite number of domains, where it is possible to apply the numerical procedure.

In some sense the limit parameters correspond to the degenerate motion of the system (1.2) and seemed to the author to be 'integrable', since the Hamiltonian H associated with these parameters describes an integrable system. In this paper we begin this investigation with the limit case $\delta \to 0, \varepsilon \to \infty$ and find that the limit system is nonintegrable. Moreover, we prove that for any value of the energy such that $\nu \neq 1$ the limit system has a hyperbolic periodic trajectory, whose invariant manifolds intersect transversally and the intersections are exponentially small.

We shall use the homoclinic invariant as a measure for separatrices splitting (see figure 2). It was introduced in [10] for area-preserving mappings. Let us modify it to be applied in our situation. Assume that an analytic Hamiltonian system with two degrees of freedom $H(\phi_1, \phi_2, p_1, p_2)$ has a hyperbolic periodic orbit $\zeta_{per}(t) = \left(\phi_1^{per}(t), \phi_2^{per}(t), p_1^{per}(t), p_2^{per}(t)\right)$ of period T, which possesses two-dimensional stable $W^s(\zeta_{per})$ and unstable $W^u(\zeta_{per})$ manifolds. It follows from the normal form theory that these manifolds can be represented in a parameteric form

$$\phi_i = \phi_i^{u,s}(t_1, t_2) \qquad p_i = p_i^{u,s}(t_1, t_2) \quad i = 1, 2$$

such that

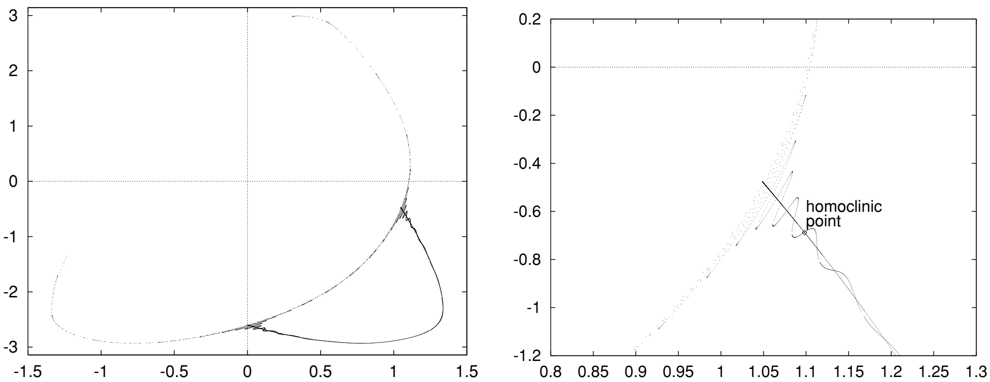


Figure 2. The splitting of separatrices, corresponding to ($\delta = 0.01$, $\varepsilon = \sqrt{5}$, $\nu = 1.16$).

1. the parametrizing functions are analytic, T-periodic in t_2;
2. for any constant t_0 the function

$$\zeta(t) = \left(\phi_1^{u,s}(t + t_0, t), \phi_2^{u,s}(t + t_0, t), p_1^{u,s}(t + t_0, t), p_2^{u,s}(t + t_0, t)\right)$$

   represents a trajectory of the system;
3. $\zeta^{u,s}(t_1, t_2) \to \zeta_{per}(t_2)$ as $t_1 \to \mp\infty$, where $\zeta^{u,s}(t_1, t_2) = \left(\phi_1^{u,s}(t_1, t_2), \phi_2^{u,s}(t_1, t_2), p_1^{u,s}(t_1, t_2), p_2^{u,s}(t_1, t_2)\right)$

The variables t_1, t_2 on the manifolds are defined uniquely up to an additive constant. This enables us to define in an invariant manner the *tangent vector* at a point $\zeta \in W^u$ (or W^s).

If $\zeta = \zeta^{u,s}(t_1, t_2)$, we set

$$\vec{e}^{u,s}(\zeta) = \frac{\partial}{\partial t_1}\zeta^{u,s}(t_1, t_2).$$

Denote the phase space of the system by M and the symplectic structure by Ω

$$\Omega = \sum_{i=1}^{2} \mathrm{d}\phi_i \wedge \mathrm{d}p_i.$$

Let us assume that the invariant manifolds of the orbit ζ_{per} intersect along a homoclinic trajectory $\zeta_h(t)$. The *homoclinic invariant* of the homoclinic trajectory ζ_h is

$$\omega(\zeta_h) = \Omega(\vec{e}^u(\zeta_h(t')), \vec{e}^s(\zeta_h(t'))). \tag{1.4}$$

The right-hand part of (1.4) is invariant with respect to the Hamiltonian flow associated with the system. So it is the same for all points of ζ_h, i.e. it does not depend on t'.

If we impose on M a Riemannian metric, then that metric induces a norm on vectors and the notion of an angle between vectors makes sense. If $\zeta_h(t')$ is a homoclinic point of a homoclinic trajectory ζ_h then the angle α between the tangent vectors to W^u and W^s at $\zeta_h(t')$ can be calculated in terms of the homoclinic invariant as

$$\sin\alpha = \frac{\omega(\zeta_h)}{||\vec{e}^u(\zeta_h(t'))||\,||\vec{e}^s(\zeta_h(t'))||}.$$

Here $||\cdot||$ stands for the norm of a vector.

To investigate the double mathematical pendulum in the limit $\delta \to 0$ we seek a solution of (1.2) in the form

$$\phi_1(t;\delta,\varepsilon,\nu) = \sum_{k=0}^{\infty}\phi_{1,k}(t;\varepsilon,\nu)\delta^k \qquad p_1(t;\delta,\varepsilon,\nu) = \sum_{k=0}^{\infty}p_{1,k}(t;\varepsilon,\nu)\delta^k$$
$$\phi_2(t;\delta,\varepsilon,\nu) = \sum_{k=0}^{\infty}\phi_{2,k}(t;\varepsilon,\nu)\delta^k \qquad p_2(t;\delta,\varepsilon,\nu) = \sum_{k=1}^{\infty}p_{2,k}(t;\varepsilon,\nu)\delta^k. \tag{1.5}$$

It follows from (1.1) that $p_{2,0} \equiv 0$. Substituting (1.5) into (1.2) and gathering the terms with the same order of δ, we obtain a recurrent system of differential equations for the coefficients in (1.5). The system of equations, corresponding to the zero order of δ, is

$$\begin{aligned}
\dot\phi_{1,0} &= p_{1,0}\\
\dot p_{1,0} &= -\sin\phi_{1,0}\\
\dot\phi_{2,0} &= \frac{p_{2,1}}{\varepsilon^2} - \frac{p_{1,0}\cos(\phi_{1,0}-\phi_{2,0})}{\varepsilon}\\
\dot p_{2,1} &= \frac{p_{1,0}p_{2,1}\sin(\phi_{1,0}-\phi_{2,0})}{\varepsilon} - \frac{\sin(2(\phi_{1,0}-\phi_{2,0}))}{2} - \varepsilon\sin\phi_{2,0}.
\end{aligned} \tag{1.6}$$

The first two equations of (1.6) are the 'pendulum equations' and their solution is

$$\phi_{1,0}(t) = \begin{cases} 2\,\mathrm{arctg}\left(\nu\dfrac{\mathrm{sn}(t-t_0,\nu)}{\mathrm{dn}(t-t_0,\nu)}\right) & \nu < 1\\ 2\,\mathrm{arctg}(\sinh(t-t_0)) \text{ or } \pi, & \nu = 1\\ 2\,\mathrm{arctg}\left(\dfrac{\mathrm{sn}\left(\nu(t-t_0),\nu^{-1}\right)}{\mathrm{cn}\left(\nu(t-t_0),\nu^{-1}\right)}\right) & \nu > 1\end{cases} \tag{1.7}$$

where $\mathrm{sn}(x,k)$, $\mathrm{cn}(x,k)$, $\mathrm{dn}(x,k)$ are the Jacobi functions of module k and t_0 is an arbitrary constant.

By elimination of $p_{2,1}$ from the third and fourth equations of (1.6), we obtain the second-order differential equation on $\phi_{2,0}$:

$$\ddot\phi_{2,0} = -\tfrac{1}{\varepsilon}\left(3\cos\phi_{1,0} + 4\nu^2 - 2\right)\sin(\phi_{2,0}-\phi_{1,0}). \tag{1.8}$$

Since the parameter ε is big we introduce a new small parameter $\rho = \frac{1}{\sqrt{\varepsilon}}$, which is more convenient in this case. Besides, we change the time variable $t \to \rho t$. Then the equation (1.8) is equivalent to the system of Hamilton equations

$$\dot x = \frac{\partial G}{\partial y} \qquad \dot y = -\frac{\partial G}{\partial x} \tag{1.9}$$

with the Hamiltonian G:

$$G = \frac{y^2}{2} - \Psi_\nu\left(\frac{t}{\rho}\right)\cos\left(x - \phi_\nu\left(\frac{t}{\rho}\right)\right)$$
$$\Psi_\nu(\tau) = \begin{cases} 6\,\mathrm{dn}^2(\tau,\nu) + 4\nu^2 - 5 & \nu < 1\\ 6\cosh^{-2}(\tau) - 1 \text{ or } -1 & \nu = 1\\ 6\,\mathrm{cn}^2(\tau,\nu) + 4\nu^2 - 5 & \nu > 1\end{cases}$$

$$\phi_\nu(\tau) = \begin{cases} 2\,\mathrm{arctg}\left(\nu \dfrac{\mathrm{sn}(\tau,\nu)}{\mathrm{dn}(\tau,\nu)}\right) & \nu < 1 \\ 2\,\mathrm{arctg}(\sinh(\tau)) \text{ or } \pi & \nu = 1 \\ 2\,\mathrm{arctg}\left(\dfrac{\mathrm{sn}(\nu\tau,\nu^{-1})}{\mathrm{cn}(\nu\tau,\nu^{-1})}\right) & \nu > 1. \end{cases} \tag{1.10}$$

Further, we consider the Hamiltonian system described by the equations (1.9) as an independent dynamical system with one and a half degrees of freedom and suppose that the parameter ρ is close to zero. In this case we study the system (1.9), find a hyperbolic periodic trajectory $(x_{per}(t), y_{per}(t))$, investigate separatrices splitting of this trajectory and prove an exponentially small asymptotic expression for the homoclinic invariant.

From these results and from the stability of hyperbolic objects under small perturbation follows:

Main theorem. *For any $\nu \neq 1$ there is a positive number $\varepsilon_0(\nu)$ such that for any $\varepsilon > \varepsilon_0(\nu)$ there exists $\delta_0(\varepsilon,\nu) > 0$ such that for $0 < \delta < \delta_0(\varepsilon,\nu)$ the system (1.2) has a periodic hyperbolic trajectory $\left(\phi_1^{per}, \phi_2^{per}, p_1^{per}, p_2^{per}\right)(t)$. The first two components of this trajectory are close to the solution of the 'pendulum' equation (1.7) and to the periodic solution $x_{per}(t)$ of the system (1.9), respectively. Moreover, the invariant manifolds associated with this trajectory intersect transversally and the intersections are exponential small with respect to the parameter ε, i.e. they are of the order of $O(e^{-const\cdot\sqrt{\varepsilon}})$.*

Remark 1. We note that the value of the parameter $\delta_0(\varepsilon,\nu)$ is very small. More precisely it is exponentially small because of the exponentially smallness of the homoclinic intersections. Thus, we prove the nonintegrability of the double mathematical pendulum for very narrow neighbourhood of the edge $(\delta = 0, \varepsilon = \infty)$ in the space of parameters.

Remark 2. As it follows from (1.7) the first two equations (1.6) have two solutions in the case $\nu = 1$. One of them $\phi_{1,0} = \pi$, $p_{1,0} = 0$ is an unstable equilibrium. It is easy to show that the corresponding Hamiltonian of the system (1.9) is of the form

$$G(x, y) = \frac{y^2}{2} - \cos x$$

and this system has the same unstable equilibrium: $x = \pi, y = 0$. Hence the system (1.6) has the unstable equilibrium $\phi_{1,0} = \pi$, $p_{1,0} = 0$, $\phi_{2,0} = \pi$, $p_{2,1} = 0$, which possesses the homoclinic trajectory $\phi_{1,0}^h = \pi$, $p_{1,0}^h = 0$, $\phi_{2,0}^h = 2\,\mathrm{arctg}(\sinh(t))$, $p_{2,1}^h = 2/\cosh t$. One can prove (see, e.g. [12]) that under small perturbation in δ the system (1.2) has a hyperbolic periodic orbit in a δ-neighbourhood of this equilibrium and its local stable and unstable manifolds are close to the mentioned homoclinic trajectory. The Poincaré–Arnold–Melnikov method predicts that the splitting of these manifolds is of the order $O(\delta)$. But the accurate asymptotic has not been obtained yet.

2. Asymptotic method

As it was mentioned the aim of the present paper is to establish the transversality of separatrices of the system (1.9) for small ρ. The standard way to study the splitting of separatrices is the Poincaré–Arnold–Melnikov method. It is usually applied to Hamiltonian systems of the form

$$K(x, y, t, \alpha) = K_0(x, y) + \alpha K_1(x, y, t)$$

where α is a small parameter and K_1 is a periodic in t. It is assumed that the unperturbed system has an unstable equilibrium $\zeta_0 = (x_0, y_0)$ and a homoclinic trajectory $\zeta_h(t) = (x_h(t), y_h(t))$.

Then the complete system has a hyperbolic periodic orbit in a α-neighbourhood of the equilibrium and the local stable and unstable manifolds associated with this periodic orbit are close to the separatrix ζ_h of the unperturbed system. In general these manifolds intersect transversally. One can take the unperturbed energy $K_0(x, y)$ as a coordinate near the separatrix $\zeta_h(t)$. Then the distance between the stable and unstable manifolds, when they reach a α-neighbourhood of a point $\zeta_h(t_0)$ for the first time, is equal to $\alpha M(t_0) + O(\alpha^2)$, where M is the Melnikov function

$$M(t_0) = \int_{-\infty}^{\infty} \{K_0, K_1\}|_{\zeta(t_0+t),t}\,\mathrm{d}t$$

where $\{\cdot\,,\cdot\}$ denotes the Poisson brackets.

If the Hamiltonian K depends on an additional parameter β, the Melnikov function may also depend on β. In particular, if the period of the perturbation K_1 is of the order of β, the Melnikov function is exponentially small with respect to β. In this case the standard Melnikov theory allows to prove the existence of the splitting for α exponentially small only.

In fact, it was established that the application of Poincaré–Arnold–Melnikov theory provides a correct prediction (but not justification) for separatrix splitting for a system with sufficiently small amplitude of perturbation. In particularly, in [5] sufficient conditions for validity of the Melnikov method were obtained in the case of weak high-frequency perturbations, that is for Hamiltonians

$$K(x, y, t/\beta; \alpha, \beta) = K_0(x, y) + \alpha K_1(x, y, t/\beta; \alpha, \beta) \tag{2.1}$$

where $|\alpha| < \alpha_0\beta^p$, $p > p_0$ and α, p, p_0 are constants.

Since the system (1.9) depends on the fast time, the splitting is exponentially small with respect to ρ [13] and cannot be detected by the classical perturbation theory. Making a time-dependent shift $(x, y) \mapsto \big(x - \varphi_\nu(t/\rho), y - \rho^{-1}\varphi'_\nu(t/\rho)\big)$ the Hamiltonian G can be rewritten in the form similar to (2.1) with $p = -2$:

$$G = \frac{y^2}{2} - a_0\cos x - \rho^{-2}\left(\sin\left(\varphi_\nu\left(\frac{t}{\rho}\right)\right)\right)x + \rho^2\left(\psi_\nu\left(\frac{t}{\rho}\right) - a_0\right)$$

where a_0 is the mean value of the function $\psi_\nu(\tau)$. One can show by using the results of [5] that for this system the limit constant $p_0 = -2$ and the Poincaré–Arnold–Melnikov method fails to predict the correct behavior of the system in the limit $\rho \to 0$. More precisely the Poincaré–Arnold–Melnikov method gives a correct order for the splitting, but one has to solve an auxiliary nonperturbative problem to obtain an actual asymptotic. Following Gelfreich we shall call this auxiliary problem a *reference system*. To investigate this situation we apply the method proposed by Lazutkin [10] for the study of the standard map and developed by Gelfreich [2]. The main idea of this method is continuation of invariant manifolds associated with a hyperbolic periodic orbit into a complex domain to study their behaviour near the singularity of a homoclinic trajectory of the averaged system, where the distance between invariant manifolds is not exponentially small. One has to note that this method can be applied to a time-periodic analytical perturbation of the intagrable system. That is why we consider the case when the parameter $\nu \neq 1$ and the Hamiltonian $G\left(x, y, \frac{t}{\rho}\right)$ is a time periodic one.

It is clear that the Hamiltonian $G(x, y, \tau)$ $\left(\tau = \frac{t}{\rho}\right)$ is a periodic one in the variable τ with period $4K(\nu)$, where $K(\nu)$ is the first kind elliptic integral of module ν:

$$K(\nu) = \begin{cases} \displaystyle\int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1 - \nu^2\sin^2\theta}} & \nu < 1 \\ \displaystyle\int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1 - \nu^{-2}\sin^2\theta}} & \nu > 1. \end{cases}$$

We change the time variable $t \to \frac{2K(\nu)}{\pi}t$. Then the Hamiltonian $\frac{2K(\nu)}{\pi}G\left(x, y, \frac{2K(\nu)}{\pi}\tau\right)$ becomes a function 2π-periodic in τ.

The method we apply has the following structure. On the first step of the method we prove the existence of a hyperbolic periodic orbit of the system (1.9) by use of the standard perturbation theory. The next stage is devoted to the invariant manifolds of the orbit. We construct suitable parameterizations for these manifolds and consider their continuation to a complex domain. In particular, we study the behaviour of the separatrices near a singularity of the averaged system, where the splitting is not exponentially small. After that we prove the existence of an integral of motion along the unstable separatrix in some complex segment, which allows us to obtain an exponentially small asymptotic formula for the distance between the invariant manifolds on the real axis. Finally we obtain an asymptotic expression for the homoclinic invariant and prove the transversality of the homoclinic intersections.

2.1. Existence of a hyperbolic periodic orbit

The main ingredient of the method is a hyperbolic periodic orbit. So let us prove the existence of such an orbit for our system.

It is well known (see, e.g. [8]) that if the Hamiltonian $G_0(x, y)$,

$$G_0(x, y) = \frac{K(\nu)}{\pi^2}\int_0^{2\pi} G\left(x, y, \frac{2K(\nu)}{\pi}\tau\right) \mathrm{d}\tau$$

the average of G in time, has a hyperbolic fixed point then the Hamiltonian G has a hyperbolic periodic orbit $(x_{per}, y_{per})(t)$ in a ρ-neighbourhood of this point.

It can be proved that

$$\frac{1}{2\pi}\int_0^{2\pi} \Psi_\nu\left(\frac{2K(\nu)}{\pi}\tau\right)\cos\left(\phi_\nu\left(\frac{2K(\nu)}{\pi}\tau\right)\right)\mathrm{d}\tau = 1. \tag{2.2}$$

We give a proof of the formula (2.2) in the appendix.

Taking into account (2.2) and that the function $\phi_\nu\left(\frac{2K(\nu)}{\pi}\tau\right)$ is odd, we obtain that the average Hamiltonian G_0 is of the form:

$$G_0(x, y) = \frac{K(\nu)}{\pi}y^2 - \frac{2K(\nu)}{\pi}\cos x. \tag{2.3}$$

The function G_0 is the Hamiltonian of the mathematical pendulum. It has a fixed hyperbolic point $x = \pi$, $y = 0$ and an associated homoclinic trajectory

$$(x_h, y_h)(t) = \left(2\,\mathrm{arctg}\left(\sinh\left(\frac{2K(\nu)}{\pi}t\right)\right), 2/\cosh\left(\frac{2K(\nu)}{\pi}t\right)\right).$$

The trajectory $(x_h, y_h)(t)$ is an analytic function in the strip $|\mathrm{Im}\ t| < \frac{\pi^2}{4K(\nu)}$ and has exactly two singularities on the boundary at $t = \pm\mathrm{i}\frac{\pi^2}{4K(\nu)}$.

It is not difficult to check that the first component of the periodic orbit (x_{per}, y_{per}) can be represented as a convergent power series

$$x_{per}\left(\frac{t}{\rho}; \rho, \nu\right) = \sum_{k=1}^{\infty} \frac{\rho^{2k}}{(2k)!} x_{per,k}\left(\frac{t}{\rho}; \nu\right)$$

where $x_{per,k}\left(\frac{t}{\rho};\nu\right)$ are 2π-periodic functions of their first argument. In particular, taking into account the Fourier expansion of the elliptic functions [1]

$$\begin{aligned}
\mathrm{sn}\left(\frac{2K(k)}{\pi}s,k\right) &= \frac{\pi}{kK(k)}\sum_{n=1}^{\infty}\frac{1}{\sinh\left[\left(n-\frac{1}{2}\right)\pi K'(k)/K(k)\right]}\sin((2n-1)s)\\
\mathrm{cn}\left(\frac{2K(k)}{\pi}s,k\right) &= \frac{\pi}{kK(k)}\sum_{n=1}^{\infty}\frac{1}{\cosh\left[\left(n-\frac{1}{2}\right)\pi K'(k)/K(k)\right]}\cos((2n-1)s)\\
\mathrm{dn}\left(\frac{2K(k)}{\pi}s,k\right) &= \frac{\pi}{2K(k)}+\frac{\pi}{K(k)}\sum_{n=1}^{\infty}\frac{1}{\cosh[n\pi K'(k)/K(k)]}\cos(2ns)
\end{aligned}\tag{2.4}$$

where $K'(k)=K(\sqrt{1-k^2})$, we get for $\nu<1$

$$x_{per,k}(\tau;\nu)=\sum_{l=1}^{\infty}r_l(\nu)\sin((2l-1)\tau)$$

$$r_l(\nu)=\left(\frac{2K(\nu)}{\pi}\right)^2\left(\left(1+4\nu^2-6\nu^2c_0(\nu)\right)g_l(\nu)-6\nu^2f_l(\nu)\right)$$

$$g_l(\nu)=\frac{2}{\nu^2(2l-1)^2\cosh\left[\left(n-\frac{1}{2}\right)\pi K'(\nu)/K(\nu)\right]}\qquad f_l(\nu)=\sum_{j\neq0}c_j(\nu)y_{l-j}(\nu)$$

$$y_l(\nu)=\begin{cases}g_l & l\geqslant1\\ -g_{-l} & l\leqslant0\end{cases}\qquad c_l(\nu)=-\frac{1}{4}\sum_{j=-\infty}^{\infty}b_j(\nu)b_{l+1-j}(\nu)\qquad b_l(\nu)=\begin{cases}a_l & l\geqslant1\\ -a_{-l} & l\leqslant0\end{cases}$$

$$a_l(\nu)=\frac{1}{\sinh\left[\left(n-\frac{1}{2}\right)\pi K'(\nu)/K(\nu)\right]}$$

and for $\nu>1$

$$x_{per,k}(\tau;\nu)=\sum_{l=1}^{\infty}q_l(\nu)\sin(2l\tau)\quad q_l(\nu)=\left(\frac{2K(\nu)}{\pi}\right)^2\left(\left(1+4\nu^2-6c_0(\nu)\right)d_l(\nu)-6h_l(\nu)\right)$$

$$d_l(\nu)=\frac{\nu^2}{l^2\cosh[l\pi K'(k)/K(k)]}\quad h_l(\nu)=\sum_{j\neq0,j\neq l}c_j(\nu)\quad z_{l-j}(\nu)z_l(\nu)=\begin{cases}d_l & l\geqslant1\\ -d_{-l} & l\leqslant-1\end{cases}.$$

2.2. Parametrization of invariant manifolds and their continuation into complex domains

The next step of the method deals with the study of the invariant manifolds associated with the hyperbolic periodic orbit obtained at the previous stage. In this subsection we represent the separatrices in a parametric form and consider their analytic continuation into the complex phase space.

Note that the system (1.9) is reversible. The involution

$$(x,y,t)\to(2\pi-x,y,-t)\tag{2.5}$$

changes the direction of the time, but does not change the equations of motion. This involution maps a periodic trajectory in a periodic trajectory and a stable manifold in an unstable manifold. The other reversing involution is defined by

$$(x,y,t)\to(-x,y,-t).$$

These symmetries of the system (1.9) imply that if a point satisfies the initial condition $x = \pi$ at $t = 0$ and belongs to the stable (unstable) manifold, then it also belongs to the unstable (stable) manifold. So it is a homoclinic point. The separatrix of the pendulum (x_h, y_h) intersects the line $x = \pi$. Consequently, the invariant manifolds of the system (1.9) also intersect this line at least for small ρ.

Let us consider the domains

$$D_1 = \{t_1 \in \mathbb{C} : \ \mathrm{Re}\, t_1 < -1\} \cup \left\{t_1 \in \mathbb{C} : -1 < \mathrm{Re}\, t_1 < 3, |\mathrm{Im}\, t_1| < \frac{\pi^2}{4K(\nu)} - 1\right\}$$

which does not contain points close to the singularities of $(x_h, y_h)(t)$and

$$D_2(\rho) = \left\{t_1 \in \mathbb{C} : -1 < \mathrm{Re}\, t_1 < 5\rho, |\mathrm{Im}\, t_1| < \frac{\pi^2}{4K(\nu)} - \rho\right\}$$

which is close to this singularity.

It follows from the general theory (see, e.g. [8]) that a periodic hyperbolic trajectory of a dynamical system with one and a half degrees of freedom possesses two-dimensional stable W^s and unstable W^u invariant manifolds in the three-dimensional extended phase space. It is convenient to represent W^u in the parametric form

$$\begin{aligned} x &= x^u(t_1, t_2; \rho, \nu) \\ y &= y^u(t_1, t_2; \rho, \nu) \\ t &= \rho t_2. \end{aligned}$$

Due to the symmetry of the system (1.9) we can define the parametrization of W^s by

$$\begin{aligned} x &= x^s(t_1, t_2; \rho, \nu) = 2\pi - x^u(-t_1, -t_2; \rho, \nu) \\ y &= y^s(t_1, t_2; \rho, \nu) = y^u(-t_1, -t_2; \rho, \nu) \qquad t = \rho t_2. \end{aligned}$$

This enables us to study only the unstable separatrix.

The following lemma states the existence of a suitable parametrization of the unstable separatrix of (x_{per}, y_{per}) and its properties.

Lemma 1. *For any $\nu \neq 1$ there is a $\rho_0(\nu) > 0$ such that for $0 < \rho < \rho_0(\nu)$ there exists a function $x^u(t_1, t_2; \rho, \nu)$, which satisfies the following conditions:*

(1) *this function is analytic in t_1 for $t_1 \in D_1 \cup D_2(\rho) \cup \bar{D}_2(\rho)$, and in t_2 for $|\mathrm{Im}\, t_2| < 1$;*
(2) *it is $2\pi\mathrm{i}$-periodic in t_1 and 2π-periodic in t_2;*
(3) *for any t_0 the function $(x, y)(t) = (x^u(t + t_0), t/\rho; \rho, \nu), y^u(t + t_0), t/\rho; \rho, \nu))$ is a trajectory of the system (1.9) and*

$$\lim_{t_1 \to -\infty} x^u(t_1, t_2; \rho, \nu) = x_{per}(t_2);$$

(4) *if $t_1 \in D_1 \cup D_2(\rho) \cup \bar{D}_2(\rho)$ and $|\mathrm{Im}\, t_2| < 1$, then the following estimate holds*

$$\left|x^u(t_1, t_2; \rho, \nu) - x_{per}(t_2) - x_h(t_1)\right| < \mathrm{const}\rho^2 \left(1 + \frac{1}{\left|t_1 - \frac{\pi^2}{4K(\nu)}\mathrm{i}\right|^2} + \frac{1}{\left|t_1 + \frac{\pi^2}{4K(\nu)}\mathrm{i}\right|^2}\right).$$

There is a unique parametrization satisfying (1)–(4) and the normalizing condition $x^u(0, 0) = \pi$.

The proof of this lemma is rather straightforward. It is easy to check that the parametrization of invariant manifolds has to satisfies the equation

$$D_t x = \frac{2K(\nu)}{\pi} y$$
$$D_t y = -\frac{2K(\nu)}{\pi} \Psi_\nu \left(\frac{2K(\nu)}{\pi} t_2\right) \cos\left(x - \phi_\nu\left(\frac{2K(\nu)}{\pi} t_2\right)\right) \tag{2.6}$$

where D_t is the differential operator defined by the formula

$$D_t = \frac{\partial}{\partial t_1} + \frac{1}{\rho}\frac{\partial}{\partial t_2}.$$

Substitute the expansion $x^u(t_1, t_2; \rho, \nu) = x_h(t_1) + u(t_1, t_2; \rho, \nu)$ into the equation (2.6). It is not difficult to show that the residual will be small. Then one has to rewrite this equation as an integral one in a suitable Banach space and use a fixed point theorem to see that the reminder $u(t_1, t_2; \rho, \nu)$ is also small. We can take as a Banach space the space of functions which are analytic in $D_1 \times \{|\mathrm{Im}\, t_2| < 1\}$ (resp. $D_2(\rho) \times \{|\mathrm{Im}\, t_2| < 1\}$) and continuous in the closure and supply it by the norm

$$||f|| = \sup_{D_1 \times \{|\mathrm{Im}\, t_2| < 1\}} |\cosh(\omega t_1) f(t_1, t_2)|$$

where ω is the Floquet exponent of the variational equations near the hyperbolic periodic trajectory x_{per}.

Then the integral operator for the domain $D_1 \times \{|\mathrm{Im}\, t_2| < 1\}$ is defined by the kernel

$$K(t, s) = \frac{\sinh(2\omega t) + 2\omega t - \sinh(2\omega s) + 2\omega s}{4\cosh(\omega t)\cosh(\omega s)}.$$

For the domain $D_2(\rho) \times \{|\mathrm{Im}\, t_2| < 1\}$ one has to apply the method developed by Gelfreich, which is based on the analysis of some difference equation (for more details see [5]).

2.3. Straightening the flow

The fact that the flow can be straightened in a neighbourhood of a segment of the unstable manifold plays the key role in the method.

Let

$$D_3(\rho) = \left\{ t_1 \in \mathbb{C} \colon |\mathrm{Re}\, t_1| \leqslant 5\rho,\ |\mathrm{Im}\, t_1| \leqslant \frac{\pi^2}{4K(\nu)} - \rho \right\}$$

and $M(r, \rho)$ be an r-neighbourhood of the corresponding segment of the unstable manifold in the complexified (extended) phase space:

$$M(r, \rho) = \{(x, y, t_2 : |x - x^u(t_1, t_2; \rho, \nu)|^2 + |y - y^u(t_1, t_2; \rho, \nu)|^2 < r^2, t_1 \in D_3(\rho), |\mathrm{Im}\, t_2| < 1\}.$$

We construct an analytical integral of the system (1.8) defined in the segment $M(r, \rho)$ and apply it to estimate the distance between separatrices.

Theorem 1. *For any* $\nu \neq 1$ *there exists* $\rho_0(\nu) > 0$*, such that for* $0 < \rho < \rho_0$ *the system (1.9) can be prolonged analytically on the domain* $M(\rho^5, \rho)$*. Moreover, there is a canonical change of variables,* $(x, y) \to (T, E)$*, defined in* $M(\rho^5, \rho)$ *and* $2\pi\rho$*-periodic in time, such that in the new variables the equations of motion take the form*

$$\frac{\mathrm{d}T}{\mathrm{d}t} = 1 \qquad \frac{\mathrm{d}E}{\mathrm{d}t} = 0$$

and

$$E\left(x^u(t_1,t_2;\rho,\nu),\,y^u(t_1,t_2;\rho,\nu),\,t_2;\rho,\nu\right)=0$$
$$T\left(x^u(t_1,t_2;\rho,\nu),\,y^u(t_1,t_2;\rho,\nu),\,t_2;\rho,\nu\right)=t_1.$$

The proof of this theorem is rather routine. So we only sketch it.

Let us consider the nonhomogeneous variational equations near the unstable manifold

$$\begin{aligned} D_t u &= \frac{2K(\nu)}{\pi}v+g_1(t_1,t_2)\\ D_t v &= \frac{2K(\nu)}{\pi}\Psi_\nu\left(\frac{2K(\nu)}{\pi}t_2\right)\cos\left(x-\phi_\nu\left(\frac{2K(\nu)}{\pi}t_2\right)\right)u+g_2(t_1,t_2)\end{aligned}\tag{2.7}$$

where the function $\vec{g}=(g_1,g_2)$ is a some known function.

Lemma 2. *Let* $\nu\neq 1$ *and* $\mu\in\left[0,\frac{1}{11}\right]$. *Then there exists a linear operator L in* $\Xi_{1+\mu}\times\Xi_{2+\mu}$ *such that* $\vec{u}=L\vec{g}$ *is a solution of (2.7), where* Ξ_μ *is the space of functions in two complex variables* t_1, t_2, *analytic in* $D_3(\rho)\times\{|\mathrm{Im}\,t_2|<1\}$ *and continuous in the closure of this domain, and the norm in* Ξ_μ *is defined by*

$$\|f\|_\mu=\sup_{D(\rho)\times\{|\mathrm{Im}t_2|<\rho\}}|\cosh^\mu(t_1)f(t_1,t_2)|.$$

Moreover, the following bound is valid

$$||L||<\mathrm{const}\log^2\rho^{-1}.$$

The proof of this lemma is similar to the proof of lemma 8 from [5] and it is based on some facts related to families of first-order linear differential equations developed in [5, 6].

We seek a family of solutions of the Hamilton equations (1.9) in the form

$$x(t_1,t_2,E;\rho,\nu)=x^u(t_1,t_2;\rho,\nu)+u(t_1,t_2,E;\rho,\nu)$$
$$y(t_1,t_2,E;\rho,\nu)=y^u(t_1,t_2;\rho,\nu)+v(t_1,t_2,E;\rho,\nu).$$

Substituting these expressions into (2.6) and separating the linear part, we obtain the system (2.7), where

$$\begin{aligned} g_1(t_1,t_2)&=0\\ g_2(t_1,t_2)&=\frac{2K(\nu)}{\pi}\Bigg(\Psi_\nu\left(\frac{2K(\nu)}{\pi}t_2\right)\sin\left(x^u(t_1,t_2)-\phi_\nu\left(\frac{2K(\nu)}{\pi}t_2\right)+u(t_1,t_2)\right)\\ &\quad-\Psi_\nu\left(\frac{2K(\nu)}{\pi}t_2\right)\sin\left(x^u(t_1,t_2)-\phi_\nu\left(\frac{2K(\nu)}{\pi}t_2\right)\right)\\ &\quad-\Psi_\nu\left(\frac{2K(\nu)}{\pi}t_2\right)\cos\left(x^u(t_1,t_2)-\phi_\nu\left(\frac{2K(\nu)}{\pi}t_2\right)\right)u(t_1,t_2)\Bigg).\end{aligned}$$

Then we can obtain the following equation

$$\vec{w}=E\vec{w}_2+L(\vec{g}(\vec{w}))\tag{2.8}$$

where $\vec{w}=(u,v)$, $\vec{g}=(g_1,g_2)$, L is the operator from lemma 2 and $\vec{w}_2(t_1,t_2)$ is the solution of the homogenuous equation (2.7), linearly independent from $\vec{w}_1=\left(\frac{\partial x^u}{\partial t_1},\frac{\partial y^u}{\partial t_1}\right)$.

It can be proved that the nonlinear operator in the right side of (2.7) is contracting in sufficiently small ball. Indeed, if $\vec{w}$ and $\vec{v}$ belong to the ball in $\Xi_{1+\mu}\times\Xi_{2+\mu}$ centred at zero and with the radius $r=\rho^{5+5\mu}$ then

$$||\vec{g}(\vec{w})||_{1+\mu,2+\mu}\leqslant\mathrm{const}\,\rho^{-2-\mu}||\vec{w}||^2_{1+\mu,2+\mu}$$
$$||\vec{g}(\vec{w})-\vec{g}(\vec{v})||_{1+\mu,2+\mu}\leqslant\mathrm{const}\,\rho^{-2-\mu}\cdot r\cdot||\vec{w}-\vec{v}||^2_{1+\mu,2+\mu}.$$

Taking into account lemma 2, we get

$$||L(\vec{g}(\vec{w}))||_{1+\mu,2+\mu} \leqslant \text{const}\,\rho^{-5-4\mu}||\vec{w}||^2_{1+\mu,2+\mu}$$
$$||L(\vec{g}(\vec{w})) - L(\vec{g}(\vec{v}))||_{1+\mu,2+\mu} \leqslant \text{const}\,\rho^{-2-\mu}\cdot r\cdot||\vec{w}-\vec{v}||^2_{1+\mu,2+\mu}.$$

Let ρ be sufficiently small, then

$$||L(\vec{g}(\vec{w}))||_{1+\mu,2+\mu} \leqslant \tfrac{r}{2}$$
$$||L(\vec{g}(\vec{w})) - L(\vec{g}(\vec{v}))||_{1+\mu,2+\mu} \leqslant \tfrac{1}{2}||\vec{w}-\vec{v}||^2_{1+\mu,2+\mu}.$$

Hence the nonlinear operator leaves the ball invariant and it is contracting provided by

$$|E|\cdot||\vec{w}_2||_{1+\mu,2+\mu} < \tfrac{r}{2}.$$

Thus, we obtain the desired solution for $|E| < E_0 = \rho^{5+6\mu}$.

Consequently, we construct the change of variables $\tilde{\Phi}_{t_2}$

$$(t_1, E) \to (x(t_1, t_2, E; \rho, \nu), y(t_1, t_2, E; \rho, \nu)).$$

This change is not canonical. But we can obtain the canonical change Φ_{t_2}, if we take $\Phi_{t_2} = \tilde{\Phi}_{t_2} \circ S^{-1}_{t_2}$, where

$$S_{t_2} : (t_1, E) \to (t_1, \tilde{E})$$
$$\tilde{E} = \int_0^E J(t_1, t_2, E')\,\mathrm{d}E'$$
$$J(t_1, t_2, E') = \det\begin{Vmatrix} \dfrac{\partial x}{\partial t_1} & \dfrac{\partial x}{\partial E} \\ \dfrac{\partial y}{\partial t_1} & \dfrac{\partial y}{\partial E}\end{Vmatrix}.$$

Under this substitution the stable manifold can be represented in the following form

$$E = \Theta(t_1 - \rho t_2; \rho, \nu), \quad T = t_1 + \mu(t_1 - \rho t_2; \rho, \nu)$$

where

$$\Theta(t_0; \rho, \nu) = E\left(x^s(t+t_0, t), y^s(t+t_0, t), t/\rho; \rho, \nu\right)$$
$$\mu(t_0; \rho, \nu) = T\left(x^s(t+t_0, t), y^s(t+t_0, t), t/\rho; \rho, \nu\right) - (t+t_0)$$

are analytic $2\pi\rho$-periodic functions. These functions are defined for $|x^s - x^u| + |y^s - y^u| < \rho^5$ what is valid for $|\mathrm{Im}\, t_1| \leqslant \pi/2 - \sigma\rho\log\rho^{-1}$ with $\sigma > 6$ and $|\mathrm{Re}\, t_1| < 5\rho$.

Taking into account the definition of the change in theorem 1, it is not difficult to see that

$$\frac{\partial E}{\partial x}\left(x^u(t_1, t_2; \rho, \nu), y^u(t_1, t_2; \rho, \nu), t_2; \rho, \nu\right) = -\frac{\partial y^u}{\partial t_1}(t_1, t_2, t_2; \rho, \nu)$$
$$\frac{\partial E}{\partial y}\left(x^u(t_1, t_2; \rho, \nu), y^u(t_1, t_2; \rho, \nu); \rho, \nu\right) = \frac{\partial x^u}{\partial t_1}(t_1, t_2; \rho, \nu).$$

Expanding T and E in the Taylor series and using first-order approximation, we have

$$\Theta(t_0; \rho, \nu) = \det\begin{Vmatrix} \dfrac{\partial x^u}{\partial t_1} & x^s - x^u \\ \frac{\partial y^u}{\partial t_1} & y^s - y^u \end{Vmatrix}(t+t_0, t/\rho) + O_2 \tag{2.8}$$

where O_2 denotes terms of the second orders in $x^s - x^u$ and t.

2.4. Reference system

It is not difficult to prove (see, e.g. [5]) that the distance $x^s - x^u$ is less than any power of ρ in the subset $\tilde{D}$ of $D_3(\rho)$

$$\tilde{D} = \left\{ t_1 \in \mathbb{C}\colon |\mathrm{Re}\, t_1| \leqslant 5\rho,\ |\mathrm{Im}\, t_1| \leqslant \frac{\pi^2}{4K(\nu)} - c,\ \text{for any } c > 0 \right\}.$$

So, to detect the splitting we have to study the separatrices for $\mathrm{Im}\, t_1 - \frac{\pi^2}{4K(\nu)} = O(\rho \log(\rho^{-1}))$, where the distance $x^s - x^u$ is no exponentially small.

We look for a Hamiltonian system, which possesses invariant manifolds, not necessarily associated with a periodic orbit, to approximate the separatrices of the system (1.9) near the singularity of the homoclinic trajectory of the averaged system. Let us introduce near the singularity of the separatrix of the averaged system new parameters and a new time by the formulae

$$s_1 = \frac{\pi}{2K(\nu)} \frac{t_1 - \pi/2}{\rho} \qquad s_2 = t_2 \qquad s = \frac{t}{\rho}.$$

Besides we make the change

$$u = \mathrm{i}x - \log \frac{\pi^2}{2\rho^2 K^2(\nu)} \qquad v = \mathrm{i}\rho \frac{2K(\nu)}{\pi} y.$$

Although this change is not canonical the equations of motion in these variables have the Hamiltonian form with

$$\tilde{G} = \frac{v^2}{2} - \Psi_\nu \left(\frac{2K(\nu)}{\pi} s \right) \mathrm{e}^{-\mathrm{i}\phi_\nu \left(\frac{2K(\nu)}{\pi} s \right)} \mathrm{e}^u + \frac{\rho^4}{4} \Psi_\nu \left(\frac{2K(\nu)}{\pi} s \right) \mathrm{e}^{\mathrm{i}\phi_\nu \left(\frac{2K(\nu)}{\pi} s \right)} \mathrm{e}^{-u}. \tag{2.9}$$

Putting $\rho = 0$, we obtain the so-called 'reference system'. This system approximates the behaviour of the system (1.9) near the singularity of the homoclinic trajectory of the averaged system. As it is shown in [2], systems with Hamiltonians

$$\frac{v^2}{2} - \gamma^2 (1 + g(s)) \cdot \mathrm{e}^u$$

where γ is a constant and g is a periodic function with zero mean value, possess two invariant manifolds. Consequently, the reference system has two invariant manifolds which can be parameterized [2] by the functions $(u_+(s_1, s_2; \nu), v_+(s_1, s_2; \nu))$ and $(u_-(s_1, s_2; \nu), v_-(s_1, s_2; \nu))$. These manifolds provide good approximation for the manifolds of the original system such that the following estimate is true

$$\det \begin{Vmatrix} \dfrac{\partial x^u}{\partial t_1} & x^s - x^u \\ \dfrac{\partial y^u}{\partial t_1} & y^s - y^u \end{Vmatrix} (t_1, t_2) = -\frac{\pi^2}{4K^2(\nu)\rho^2} \det \begin{Vmatrix} \dfrac{\partial u_-}{\partial s_1} & u_+ - u_- \\ \dfrac{\partial v_-}{\partial t_1} & v_+ - v_- \end{Vmatrix} (s_1, s_2)\, (1 + O(\rho))$$

which is valid on the intersection of the line $\mathrm{Im}\, t_1 = \pi/2 - \sigma\rho \log(\rho^{-1})$ with the domain $D_3(\rho)$ for any $\sigma > 1$.

Lemma 3. [2] *For any $\nu \neq 0$ there is a constant $\Theta_1(\nu)$ such that in the sector $-\pi + c < \arg s_1 < -c$*

$$\det \begin{Vmatrix} \dfrac{\partial u_-}{\partial s_1} & u_+ - u_- \\ \dfrac{\partial v_-}{\partial t_1} & v_+ - v_- \end{Vmatrix} (s_1, s_2) = \Theta_1(\nu) \exp(-\mathrm{i}(s_1 - s_2)) + O(\exp(-(2 - c')|\mathrm{Im}\, s_1|))$$

for any $c' > 0$, $\mathrm{Im}\, s_1 < -A$, $A > 0$. The constant in the estimate term depends on c, c', A and on the parameter ν.

Then we have

$$\Theta(t_0; \rho, \nu) = -2\frac{\pi^2}{4K^2(\nu)\rho^2}\Theta_1(\nu)\exp\left(\frac{\pi}{2K(\nu)}\frac{-\pi/2 - \mathrm{i}t_0}{\rho}\right)(1 + O(\rho))$$

on the line $\mathrm{Im}\, t_0 = \pi/2 - \sigma\rho\log(\rho^{-1})$. Since the function Θ is a real analytic one, we obtain that

$$\Theta(t_0; \rho, \nu) = -2\frac{\pi^2}{4K^2(\nu)\rho^2}|\Theta_1(\nu)|\exp\left(\frac{-\pi^2}{4K(\nu)\rho}\right)\cos\left(\frac{\pi t_0}{2K(\nu)\rho} - \arg\Theta_1\right)(1 + O(\rho)) \tag{2.10}$$

inside the strip $|\mathrm{Im}\, t_0| \leqslant \pi/2 - \sigma\rho\log(\rho^{-1})$.

Finally, we have to mention that since the function $\Theta(t_0; \rho, \nu)$ can be considered as a measure of the distance between separatrices, its zeros correspond to homoclinic points.

2.5. Asymptotic formula for the homoclinic invariant

In this subsection we calculate the homoclinic invariant for the reduced system. One should note that the homoclinic invariant was defined by (1.4) for autonomous Hamiltonian systems with two degree of freedom while the system (1.9) is a system with one and a half degrees of freedom. Nevertheless, it is well known that such a system can be rewritten as an autonomous Hamiltonian system with two degree of freedom if we take the time variable as a new coordinate and the Hamiltonian as a conjugate momentum. Moreover, it follows from the definition of the tangent vectors $\vec{e}^{u,s}$ that in this case the homoclinic invariant of some homoclinic trajectory $\zeta_h(t) = (x_h(t), y_h(t), t, G(x_h(t), y_h(t), t))$ has the following form

$$\omega(\zeta_h) = \Omega_{red}(\vec{e}^u(\zeta_h(t'), \vec{e}^s(\zeta_h(t'))$$

where $\Omega_{red} = \mathrm{d}y \wedge \mathrm{d}x$.

It is easy to show (see [6]) that the homoclinic invariant ω_h can be represented in terms of the function $\Theta(t_0; \rho, \nu)$

$$\omega_h(\rho, \nu) = \frac{\mathrm{d}\Theta}{\mathrm{d}t_0}(t_{0k}; \rho, \nu)$$

where t_{0k} is a zero of the function Θ.

Taking into account (2.10), we get

$$\omega_h(\rho, \nu) = \frac{\pi^3}{4K^3(\nu)\rho^3}|\Theta_1(\nu)|\exp\frac{-\pi^2}{4K(\nu)\rho}(1 + O(\rho)). \tag{2.11}$$

It will be shown in the following section the correction factor is nonzero. Hence the homoclinic invariant (2.11) does not vanish and the separatrices of the hyperbolic periodic orbit described in subsection 2.1 intersect transversally. It implies nonintegrability of the reduced system (1.9). Moreover, since the full system (1.2) is a small perturbation of (1.9) as the parameter δ tends to zero, one can get by using the standard perturbation theory that there exists a positive constant $\delta_0(\varepsilon, \nu)$ such that the full system is nonintegrable too for any $\delta < \delta_0(\varepsilon, \nu)$. It follows from the exponential smallness of the homoclinic intersections for the reduced system that the constant $\delta_0(\varepsilon, \nu)$ is exponentially small with respect to the parameter $1/\sqrt{\varepsilon}$.

3. Numerical calculations

Note that the asymptotic expression for the homoclinic invariant includes the factor $\Theta_1(\nu)$. It is a remarkable part of the formula. As it was mentioned above the Poincaré–Arnold–Melnikov method predicts the formula (2.11) up to this correction factor, which describes the splitting of the complex invariant manifold associated with the reference system. A set of similar constants appeared in the study of some other Hamiltonian systems [2] and area-preserving maps [3]. Nevertheless, its nature is not completely clear. The rest of the paper is devoted to calculation of the constant $\Theta_1(\nu)$.

For this purpose we use the following method proposed in [2]. Let us seek for the invariant manifold of the reference system in the form

$$u(s_1, s_2; \nu) = -\log\frac{s_1^2}{2} + \sum_{n\geqslant 2}\frac{a_n(s_2;\nu)}{n!s_1^n} \tag{3.1}$$

where the functions $a_k(s_2;\nu)$ are 2π-periodic with respect to s_2. The parameters $s_1,\ s_2$ are chosen in such a way that $u(s+s_0, s;\nu)$ is a solution of the following equation

$$u'' = \Psi_\nu\left(\frac{2K(\nu)}{\pi}s\right)\mathrm{e}^{-\mathrm{i}\phi_\nu\left(\frac{2K(\nu)}{\pi}s\right)}\mathrm{e}^u. \tag{3.2}$$

It is to be noted that this equation is equivalent to the equations of motion of the reference system.

It follows from [2] that there are two invariant manifolds which have asymptotic expansion (3.1) in the sectors $c < \arg s_1 < 2\pi - c$ and $-\pi + c < \arg s_1 < \pi - c$, respectively.

Substituting (3.1) into (3.2) and collecting the terms with the same power of s_1^{-1}, we obtain the equations on the functions a_k

$$\begin{aligned}
&2 + \frac{a_2''(s_2;\nu)}{2!} = 2\Psi_\nu\left(\frac{2K(\nu)}{\pi}s_2\right)\mathrm{e}^{-\mathrm{i}\phi_\nu\left(\frac{2K(\nu)}{\pi}s_2\right)} - \frac{2a_2'(s_2;\nu)}{1!} + \frac{a_3''(s_2;\nu)}{3!} = 0\\
&\frac{(n-1)a_{n-2}(s_2;\nu)}{(n-3)!} - \frac{2a_{n-1}'(s_2;\nu)}{(n-2)!} + \frac{a_n''(s_2;\nu)}{n!}\\
&\qquad = 2\Psi_\mu\left(\frac{2K(\nu)}{\pi}s_2\right)\mathrm{e}^{-\mathrm{i}\phi_\mu\left(\frac{2K(\nu)}{\pi}s_2\right)}\frac{Y_{n-2}(a_1,\ldots,a_{n-2})(s_2)}{(n-2)!} \qquad n\geqslant 4.
\end{aligned} \tag{3.3}$$

Here Y_n are Bell polynomials defined by the recurrent formula

$$\begin{aligned}
&Y_0 = 1\\
&Y_n(a_1,\ldots,a_2) = \sum_{k=0}^{n-1} C_{n-1}^k Y_k(a_1,\ldots,a_k)a_{n-k} \qquad n\geqslant 1.
\end{aligned}$$

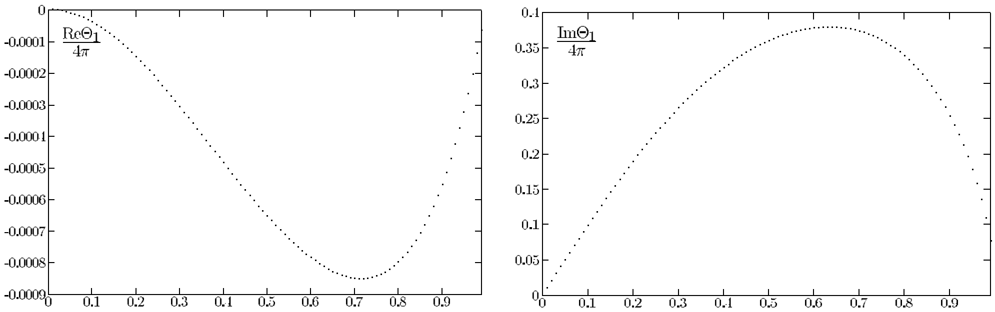
It is to be noted that $Y_1 = a_1 \equiv 0$. We can apply the Fourier method to solve the equations (3.3). Denoting the Fourier transformation of the function $g = \left(\Psi_\nu \mathrm{e}^{-\mathrm{i}\phi_\nu} - 1\right)$ by $F[g]$, we have

$$\begin{aligned}
&(n-1)(n-2)F[a_{n-2}](k) - 2ikF[a_{n-1}](k) - \frac{k^2}{n(n-1)}F[a_n](k)\\
&\qquad = 2(F[g+1]*F[Y_{n-2}])(k) \qquad k\in\mathbb{Z} \qquad k\neq 0,\\
&(n^2-3n)F[a_{n-2}](0)\\
&\qquad = 2(F[g]*F[a_{n-2}])(0) + 2(F[g+1]*\sum_{l=2}^{n-3}C_{n-3}^l(F[Y_l]*F[a_{n-2-l}]))(0).
\end{aligned} \tag{3.4}$$

where the sign '$*$' denotes the convolution.

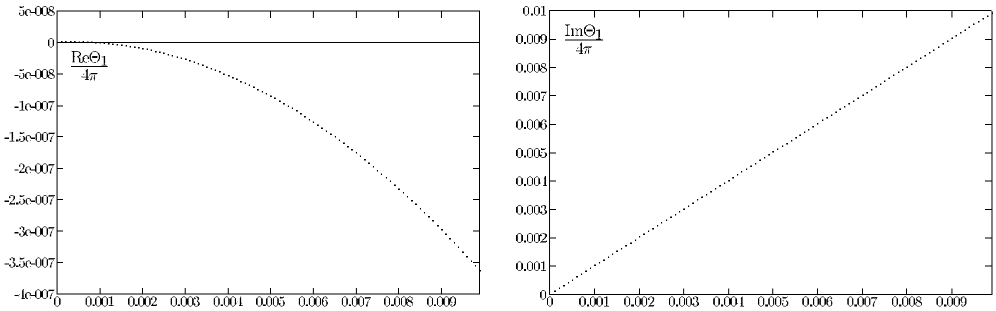
Taking into account (2.4) and solving the system (3.4), we obtain the coefficients of the expansion (3.1). Then we calculate

$$u(s_1, s_2; \nu) = -\log\frac{s_1^2}{2} + \sum_{n\geqslant 2}\frac{a_n(s_2;\nu)}{n!s_1^n}$$



(a) (b)

Figure 3. The dependence of $\mathrm{Re}\Theta_1$ (a) and $\mathrm{Im}\Theta_1$ (b) on ν, when $\nu < 1$.



(a) (b)

Figure 4. The dependence of $\mathrm{Re}\Theta_1$ (a) and $\mathrm{Im}\Theta_1$ (b) on ν, when $\nu < 1$ and ν is sufficiently small.

$$
\begin{aligned}
v(s_1, s_2; \nu) &= -\frac{2}{s_1} + \sum_{n\geqslant 2} \frac{a_n'(s_2;\nu)}{n!s_1^n} - \sum_{n\geqslant 2} \frac{a_n(s_2;\nu)}{(n-1)!s_1^{n+1}} \\
\frac{\partial u}{\partial s_1}(s_1, s_2; \nu) &= -\frac{2}{s_1} - \sum_{n\geqslant 2} \frac{a_n(s_2;\nu)}{(n-1)!s_1^{n+1}} \\
\frac{\partial v}{\partial s_1}(s_1, s_2; \nu) &= \frac{2}{s_1^2} - \sum_{n\geqslant 2} \frac{a_n'(s_2;\nu)}{(n-1)!s_1^{n+1}} + \sum_{n\geqslant 2} \frac{(n+1)a_n(s_2;\nu)}{(n-1)!s_1^{n+2}}
\end{aligned}
\tag{3.5}
$$

using the first eight terms of the series.

Taking these values as initial conditions, we integrate the equations of motion and the variational equations

$$
\begin{aligned}
&\frac{\mathrm{d}u}{\mathrm{d}s} = v \qquad \frac{\mathrm{d}v}{\mathrm{d}s} = \Psi_\nu\left(\frac{2K(\nu)}{\pi}s\right)\mathrm{e}^{-\mathrm{i}\phi_\nu\left(\frac{2K(\nu)}{\pi}s\right)}\mathrm{e}^u \\
&\frac{\mathrm{d}\frac{\partial u}{\partial s_1}}{\mathrm{d}s} = \frac{\partial v}{\partial s_1} \qquad \frac{\mathrm{d}\frac{\partial v}{\partial s_1}}{\mathrm{d}s} = \Psi_\nu\left(\frac{2K(\nu)}{\pi}s\right)\mathrm{e}^{-\mathrm{i}\phi_\nu}\left(\frac{2K(\nu)}{\pi}s\right)\mathrm{e}^u\frac{\partial u}{\partial s_1}.
\end{aligned}
\tag{3.6}
$$

Starting at the point $(s_1 - R, s_2 - R)$ with sufficiently big constant R, we calculate the point on the unstable manifold corresponding to the values of parameters (s_1, s_2) and the tangent vector to the unstable manifold at this point. Starting at $(s_1 + R, s_2 + R)$, we obtain the point and the tangent vector corresponding to the stable manifold. We verified the values

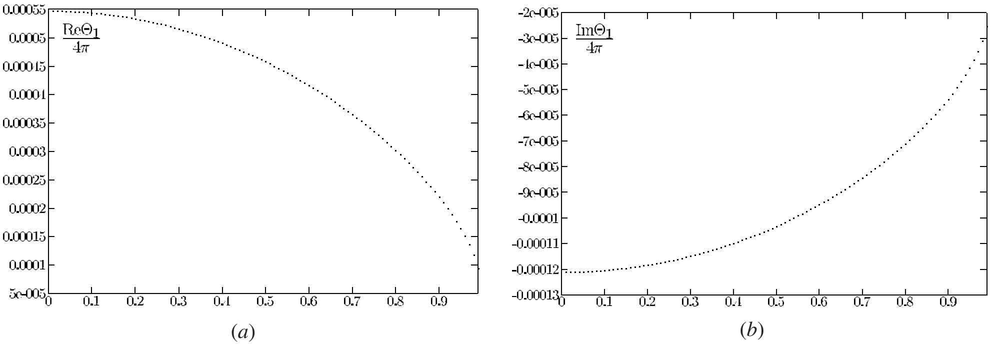


Figure 5. The dependence of $\mathrm{Re}\Theta_1$ (*a*) and $\mathrm{Im}\Theta_1$ (*b*) on $\frac{1}{\nu}$, when $\nu > 1$.

of R from 200 to 1000 and found out that the result is independent on this choice. Then we calculate the following approximation of $\Theta_1(\nu)$

$$\tilde{\Theta}_1(s_1, s_2; \nu) = \exp(\mathrm{i}(s_1 - s_2)) \det \begin{Vmatrix} \dfrac{\partial u_-}{\partial s_1} & u_+ - u_- \\ \dfrac{\partial v_-}{\partial t_1} & v_+ - v_- \end{Vmatrix} (s_1, s_2; \nu). \tag{3.7}$$

As it follows from lemma 3

$$\tilde{\Theta}_1(s_1, s_2; \nu) - \Theta_1(\nu) = O(\exp(-(1 - c')|\mathrm{Im}\, s_1|)).$$

We fixed $s_1 = -17i, s_2 = 0, R = 600$ and obtained the dependence of $\tilde{\Theta}_1$ on the parameter ν. These results are presented in figures 3–5.

It is easy to see that the function $g = g\left(\frac{2K(\nu)}{\pi}s;\, \nu\right)$ is an analytic one with respect to ν at zero for all s and $g\left(\frac{2K(\nu)}{\pi}s;\, 0\right) \equiv 0$. Then the following lemma holds (see [2]).

Lemma 4. *There is a constant* $\nu_0 > 0$ *such that for* $0 < \nu < \nu_0$ *the constant* $\Theta_1(\nu)$ *has the following approximation*

$$\Theta_1(\nu) = -4\pi \mathrm{i}\nu \frac{\mathrm{d}g_1(\nu)}{\mathrm{d}\nu}(0)(1 + O(\nu))$$

where $g_1(\nu)$ *is the first coefficient of the Fourier series* $g(s; \nu) = \sum_{k=-\infty}^{\infty} g_k(\nu)\mathrm{e}^{\mathrm{i}ks}$.

Using expansions (2.4), it is not difficult to obtain that

$$\frac{\mathrm{d}g_1(\nu)}{\mathrm{d}\nu}(0) = -1.$$

Finally, we have

$$\Theta_1(\nu) = 4\pi \mathrm{i}\nu(1 + O(\nu)).$$

It is to be noted that the numerical calculations are in a good agreement with this formula (see figures 4(a) and (b)).

4. Conclusions

In the paper we prove the existence of exponentially small transversal homoclinic intersections for the double mathematical pendulum in the limit when the ratio of pendulums masses close to zero and the ratio of pendulum lengths tend to infinity. A preliminary analysis of the system shows that exponentially small homoclinic intersections may occur in other limit cases. For example, when the parameter ν is small or big and the other two parameters are of the order of $O(1)$. But in these cases the situation is different, since the limit systems have no hyperbolic periodic orbits or unstable equilibria. Nevertheless, after small modification the method used in this paper can be applied to the described situation too. One should note the adaptability of the method. It is clear enough that the first two steps of the method dealing with the construction of a hyperbolic periodic orbit, its separatrices and justification of corresponding asymptotics can be done for a general system. The coordinate system from step 3 is the central and most complicated part of the method. For a general system one can get a straightened flow theorem [5], but it is not enough for proving the asymptotic formulae. For several other examples the complete proofs were obtained [2], [7], but as in the present paper those proofs depend on the behaviour of the invariant manifolds and the solutions of the normal variation equations near the singularity of the homoclinic trajectory of the averaged system. The substitution used on step 4 is not general, but it is suitable to obtain a reference system for Hamiltonians which are trigonometrical polinomials with respect to the space variables. The symmetry of the double mathematical pendulum is useful, but it is not essential, since we can construct all ingredients for the unstable as for the stable separatrix. Besides the method can be modified to be applicable to non-Hamiltonian systems. Of course, the change of variables in the straightened flow theorem is not canonical in this case. The proof can be done like in [6], where the case of near identity maps was considered.

Appendix

First we consider the case $\nu > 1$. Then the integral from the left-hand part of (2.2) is equal to the following integral

$$I = \frac{1}{4K(\nu)} \int_0^{4K(\nu)} (6\mathrm{cn}^2(\tau, \nu^{-1}) + 4\nu^2 - 5)(2\mathrm{cn}^2(\tau, \nu^{-1}) - 1)\, \mathrm{d}\tau$$
$$= \frac{3}{K(\nu)} \int_0^{4K(\nu)} \mathrm{cn}^4(\tau, \nu^{-1})\, \mathrm{d}\tau + \frac{2\nu^2 - 4}{K(\nu)} \int_0^{4K(\nu)} \mathrm{cn}^2(\tau, \nu^{-1})\, \mathrm{d}\tau + 5 - 4\nu^2. \tag{A1}$$

It is well known (see, e.g. [1]) that the Jacobi functions satisfy the following conditions

$$\begin{aligned} \mathrm{cn}^2(\tau, k) + \mathrm{sn}^2(\tau, k) &= 1 \\ \mathrm{dn}^2(\tau, k) + k^2\mathrm{sn}^2(\tau, k) &= 1 \end{aligned} \tag{A2}$$

and

$$\begin{aligned} \mathrm{sn}'(\tau, k) &= \mathrm{cn}(\tau, k)\, \mathrm{dn}(\tau, k) \\ \mathrm{cn}'(\tau, k) &= -\mathrm{sn}(\tau, k)\, \mathrm{dn}(\tau, k) \\ \mathrm{dn}'(\tau, k) &= -k^2\mathrm{sn}(\tau, k)\mathrm{cn}(\tau, k). \end{aligned} \tag{A3}$$

Using (A2), we have

$$S_1 = \int_0^{4K(k)} \mathrm{cn}^2(\tau, \nu^{-1})\, \mathrm{d}\tau = \frac{1}{k^2} E(4K(k)) - \frac{k'^2}{k^2} \tag{A4}$$

where $E(z) = \int_0^z \mathrm{dn}^2(\tau, k)\, \mathrm{d}\tau$, $k'^2 = 1 - k^2$.

We consider the following integral

$$S_2 = \int_0^{4K(k)} \mathrm{cn}^4(\tau, \nu^{-1})\, \mathrm{d}\tau.$$

Combining the formulae (A2) and (A3), we get three different expressions for S_2:

$$S_2 = S_1 - \frac{1}{k^2}\int_0^{4K(k)} \mathrm{sn}'^2(\tau, \nu^{-1})\,\mathrm{d}\tau + \frac{1}{k^2}\int_0^{4K(k)} \mathrm{cn}'^2(\tau, \nu^{-1})\,\mathrm{d}\tau \tag{A5}$$

$$S_2 = \frac{1}{k^2}\int_0^{4K(k)} \mathrm{sn}'^2(\tau, \nu^{-1})\,\mathrm{d}\tau - \frac{k'^2}{k^2}S_1 \tag{A6}$$

$$S_2 = S_1 + \frac{k'^2}{k^2}\int_0^{4K(k)} \mathrm{sn}^2(\tau, \nu^{-1})\,\mathrm{d}\tau - \frac{1}{k^2}\int_0^{4K(k)} \mathrm{cn}'^2(\tau, \nu^{-1})\,\mathrm{d}\tau. \tag{A7}$$

Summarizing (A5)–(A7) and taking into account that $\int_0^{4K(k)} \mathrm{sn}^2(\tau, \nu^{-1})\,\mathrm{d}\tau = 4K(k) - S_1$, we obtain

$$S_2 = \frac{2}{3}\left(1 - \frac{k'^2}{k^2}\right)S_1 + \frac{4}{3}\frac{k'^2}{k^2}K(k). \tag{A8}$$

It can be proved [1] that $E(4K(k)) = 4E_1(k)$, where $E_1(k) = \int_0^{\pi/2}\sqrt{1 - k^2\sin^2(\tau)}\,\mathrm{d}\tau$. Taking this into account and substituting (A8) into (A1), we prove the formula (2.2).

In the case $\nu < 1$ the integral from the left-hand part of (2.2) is equal to the following integral

$$I = \frac{1}{4K(\nu)}\int_0^{4K(\nu)} (6\mathrm{dn}^2(\tau, \nu) + 4\nu^2 - 5)(2\mathrm{dn}^2(\tau, \nu) - 1)\,\mathrm{d}\tau. \tag{A9}$$

Using (A2) we have that

$$I = \frac{3\nu^4}{K(\nu)}\int_0^{4K(\nu)} \mathrm{cn}^4(\tau, \nu)\,\mathrm{d}\tau + \frac{2\nu^2 - 4\nu^4}{K(\nu)}\int_0^{4K(\nu)} \mathrm{cn}^2(\tau, \nu)\mathrm{d}\tau + (1 - 2\nu^2)^2. \tag{A10}$$

Substituting (A4) and (A8) into (A10), we prove the formula (2.2) for $\nu < 1$.

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